Constrained knots in lens spaces

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Sketch

Constrained knots K:

- $(1,1)$ knots in $L(p, q^{-1})$;
- generalization of 2-bridge knots $\mathfrak{b}(u,v)$;
- parameterized by $C(p,q,l,u,v)$ (Y. '20);
- have a complete classification (Main theorem, Y. '20);
- whose $\widehat{HFK}$ and KHI are determined by Alexander polynomial, Moreover, $\widehat{HFK}(K) \cong KHI(K)$ (Li and Y. '20, '21, Baldwin, Li, and Y. '20);
- whose complements include many simple hyperbolic manifolds (Y. '20).

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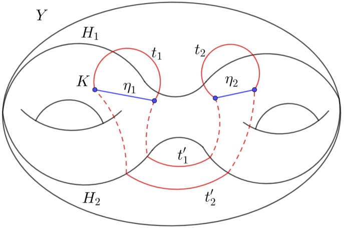
(g, b) knots

Definition

A knot $K \subset Y$ is a (g, b) (g-**genus** b-**bridge**) knot if Y admits a Heegaard splitting $Y = H_1 \cup_{\Sigma_g} H_2$ such that $K \cap H_i$ consists of b **trivial** arcs.

Remark

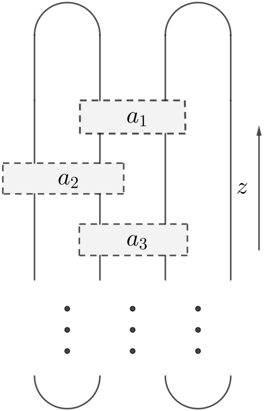
Arcs $t_1, \ldots, t_b$ are trivial in H if there exist disks $D_1, \ldots, D_b \subset H$ such that $\partial D_i = t_i \cup \eta_i$, $\eta_i \subset \partial H$, and $D_i \cap t_j = \emptyset$ for $i \neq j$.



Note: (g, b) knots are also $(g+1, b-1)$ knots.

2-bridge knots

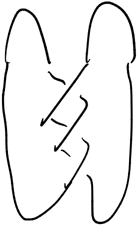
$(0,2)$ knots are called **2-bridge knots** (also **rational knots**), denoted by $\mathfrak{b}(a,b)$, where a is odd, $b \in \mathbb{Z}$, and $\gcd(a,b)=1$.



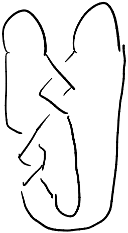
Expand b/a as continued fraction:

$$\frac{b}{a} = \frac{1}{a_1 - \frac{1}{a_2 - \cdots}}$$

$\frac{1}{3}$



$$\frac{2}{5} = \frac{1}{2 - \frac{1}{-2}}$$



2-bridge knots

Proposition (Classification, Schubert '56)

- 2-bridge knots $\mathfrak{b}(a_1, b_1)$ and $\mathfrak{b}(a_2, b_2)$ are equivalent if and only if

$$a_1 = a_2 = a \text{ and } b_1 \equiv b_2^{\pm 1} \pmod{a}.$$

- $\mathfrak{b}(a, -b)$ is the mirror knot of $\mathfrak{b}(a, b)$.

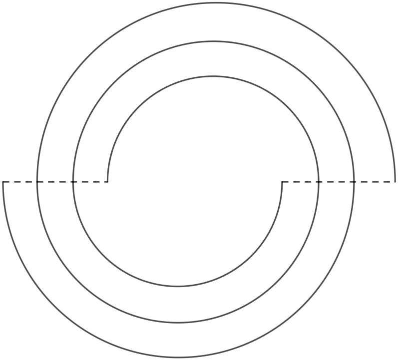
Remark

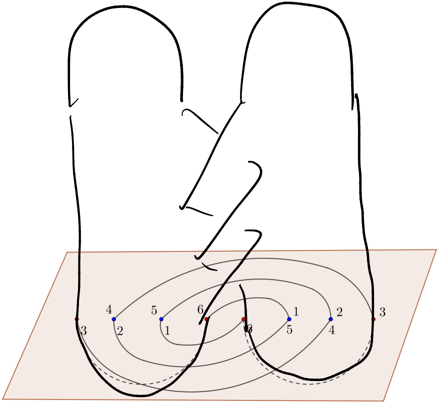
The double branched cover over $\mathfrak{b}(a, b)$ is the lens space $L(a, b)$.

2-bridge knots

A 2-bridge knot $\mathfrak{b}(a, b)$ admits another canonical presentation known as the **Schubert normal form**.

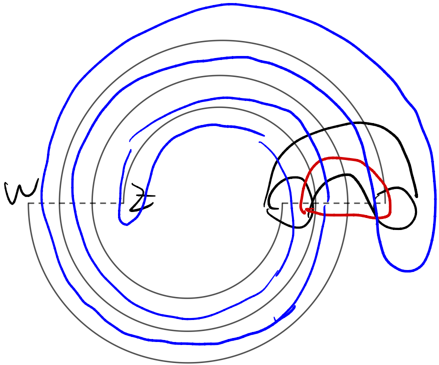
b(3,1)

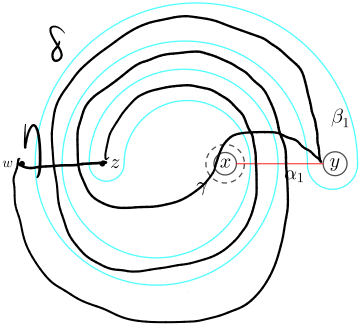




2-bridge knots

A doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$ defines a knot K. Let $\eta \subset \Sigma - \alpha$ and $\delta \subset \Sigma - \beta$ be arcs connecting z and w. Push η into α-handlebody to obtain η'. Similarly define δ' in β-handlebody. Define $K = \eta' \cup \delta'$.





$(1,1)$ knots

Definition

A $(1,1)$ knot has a doubly-pointed Heegaad diagram $(\Sigma, \alpha, \beta, z, w)$ with $\Sigma \cong T^2$, called a $(1,1)$ **diagram**.

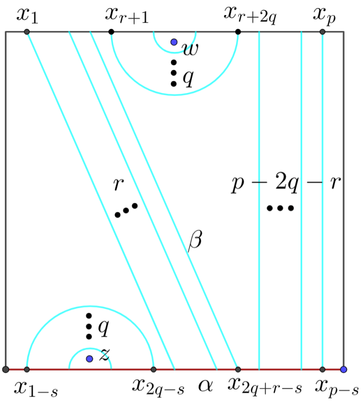
Remark

The ambient 3-manifold Y of a $(1,1)$ knot is either S^3, a lens space $L(p,q)$, or $S^1 \times S^2$. In this talk, we only consider $Y = S^3$ or $Y = L(p,q)$.

$(1,1)$ knots

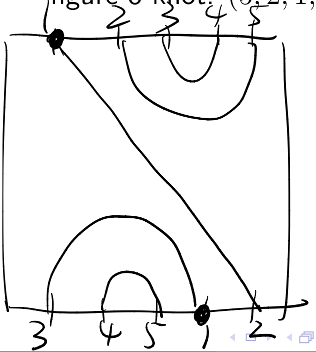
Proposition (Parameterization, Goda, Matsuda, and Morifuji '05)

$(1,1)$ diagrams are parameterized by $p, q, r, s \in \mathbb{N}$ with $2q + r \leq p$ and $s < p$.



Not good parameterization:

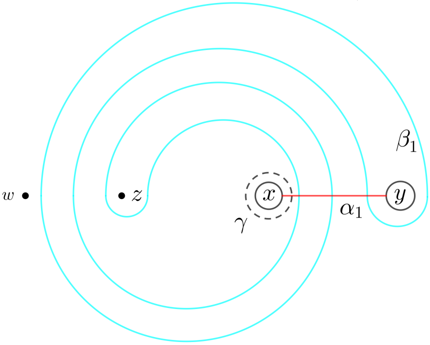
figure-8 knot: $(5,2,1,3)$ $(5,2,1,0)$

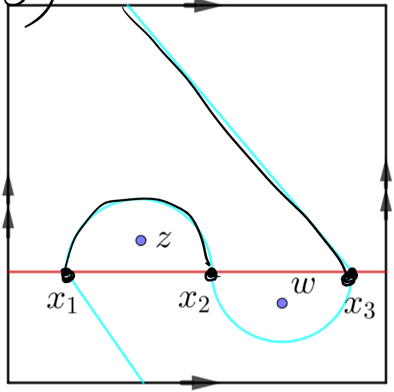


Fact

For $a > 2b > 0$, the 2-bridge knot $\mathfrak{b}(a,b)$ is the $(1,1)$ knot $(a, b, a-2b, 0)$.

b(3,1) (3,1,1,0)





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Constrained knots

Theorem (Parameterization, Y. '20)

Any constrained knot can be represented by $C(p,q,l,u,v)$, where $p>0$, $q\in[1,p-1]$, $l\in[1,p]$, $u>0$, $v\in[0,u-1]$, u is odd, $\gcd(p,q)=\gcd(u,v)=1$.

Theorem (Classification, Y. '20)

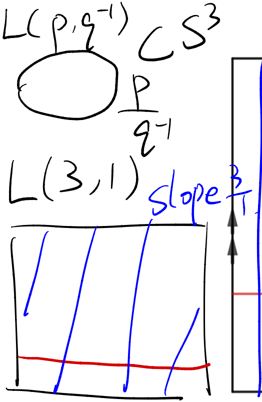
For $K_i=C(p_i,q_i,l_i,u_i,v_i)(i=1,2)$ with $p_i>0,l_i>1$ and $u_i>2v_i>0$, they represent the same knot if and only if

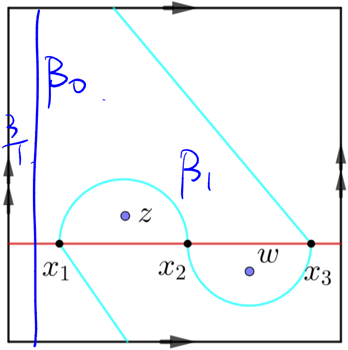
$$p_1=p_2=p,\quad q_1q_2\equiv 1 \pmod p,$$

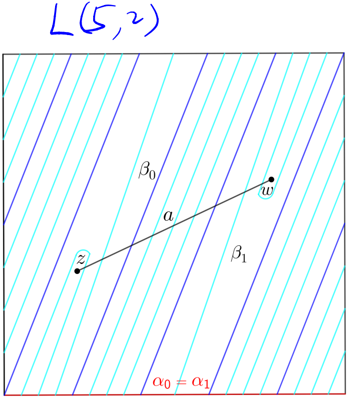
$$l_1,l_2\in\{2,p\},\quad (l_1,u_1,v_1)=(l_2,u_2,v_2).$$

Constrained knots

For a lens space $L(p, q^{-1})$, let α_0 and β_0 be two curves on T^2 with slopes 0 and p/q^{-1}. Let $\alpha_1 = \alpha_0$ and let β_1 be a curve with $\beta_1 \cap \beta_0 = \emptyset$. Set $z, w \in T^2 - \alpha_0 \cup \beta_0 \cup \beta_1$. Define a constrained knot by $(T^2, \alpha_1, \beta_1, z, w)$.

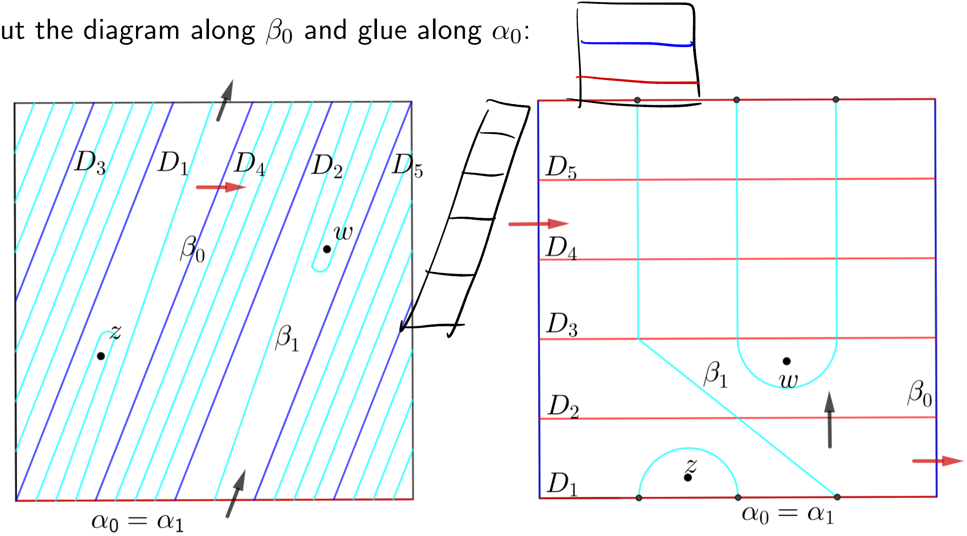






Constrained knots

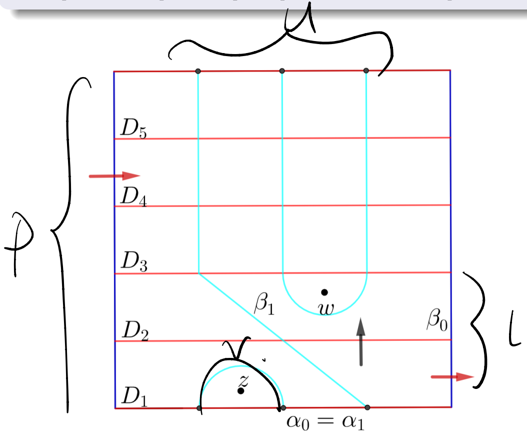
Cut the diagram along β_0 and glue along α_0:



Constrained knots

Theorem (Parameterization, Y. '20)

Any constrained knot can be represented by $C(p,q,l,u,v)$, where $p>0$, $q\in[1,p-1]$, $l\in[1,p]$, $u>0$, $v\in[0,u-1]$, u is odd, $\gcd(p,q)=\gcd(u,v)=1$.



$C(5,3,2,3,1)$.

$p=5=$ number of domains

$q=3: D_1\to D_{1+q}$

$l=2: z\in D_1, w\in D_l$

$u=3=|\beta_1\cap\{\text{subarc of } \alpha_1\}|$

$v=1=$ number of rainbows

Constrained knots

Theorem (Classification, Y. '20)

For $K_i = C(p_i, q_i, l_i, u_i, v_i)(i = 1, 2)$ with $p_i > 0, l_i > 1$ and $u_i > 2v_i > 0$, they represent the same knot if and only if

$$p_1 = p_2 = p, \quad q_1 q_2 \equiv 1 \pmod{p},$$

$$l_1, l_2 \in \{2, p\}, \quad (l_1, u_1, v_1) = (l_2, u_2, v_2).$$

Remark

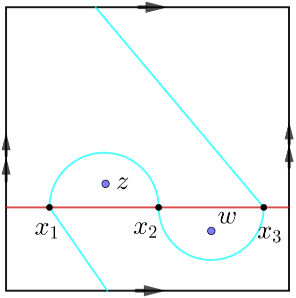
The red conditions can be explained by the following facts.

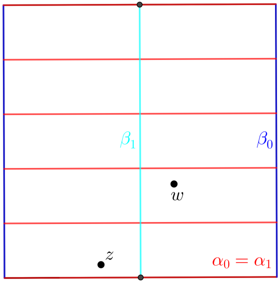
Constrained knots

Fact

(p, q)

- $C(1,0,1,u,v)$ is the 2-bridge knot $\mathfrak{b}(u,v)$;
- $C(p,q,l,1,0)$ consists of simple knots in lens spaces studied by Rasmussen, Hedden, *et al.* (related to Berge's conjecture).

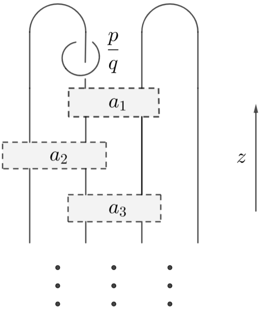




Constrained knots

Fact

- $C(1,0,1,u,v)$ is the 2-bridge knot $\mathfrak{b}(u,v)$;
- $C(p,q,l,1,0)$ consists of simple knots;
- $C(p,q,1,u,v)$ is a connected sum of a 2-bridge knot and a core knot in a lens space.



Constrained knots

Fact

- $C(1,0,1,u,v)$ is the 2-bridge knot $\mathfrak{b}(u,v)$;
- $C(p,q,l,1,0)$ consists of simple knots;
- $C(p,q,1,u,v)$ is a connected sum of a 2-bridge knot and a core knot in a lens space;
- $C(p,-q,l,u,-v)$ is the mirror knot of $C(p,q,l,u,v)$.

Remark

We only need to consider $(p,q)\neq(1,0)$, $(u,v)\neq(1,0)$, $l\neq 1$, $u>2v>0$.

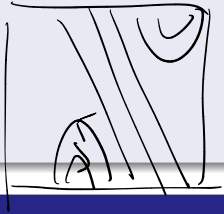
Constrained knots

Theorem (Classification, Y. '20)

For $K_i = C(p_i, q_i, l_i, u_i, v_i)(i = 1, 2)$ with $p_i > 0, l_i > 1$ and $u_i > 2v_i > 0$, they represent the same knot if and only if

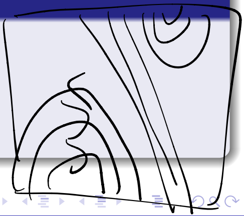
$$p_1 = p_2 = p, \quad q_1 q_2 \equiv 1 \pmod p,$$

$$l_1, l_2 \in \{2, p\}, \quad (l_1, u_1, v_1) = (l_2, u_2, v_2).$$



Remark

- $C(5, 3, l, 3, 1) \cong C(5, 2, l, 3, 1)$ for $l = 2, 5$;
- $C(5, 3, l, 3, 1) \ncong C(5, 2, l, 3, 1)$ for $l = 3, 4$;
- There is no known classification of $(1, 1)$ knots.

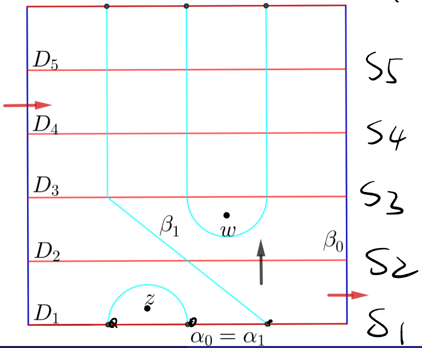


Constrained knots

Idea of necessary part: compute knot Floer homology $\widehat{HFK}$ defined by Oszváth and Szabó, Rasmussen. For $K = C(p,q,l,u,v) \in Y = L(p,q^{-1})$,

$$\widehat{HFK}(Y,K) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \widehat{HFK}(Y,K,\mathfrak{s}) \cong \mathbb{Z}^{|\alpha_1\cap\beta_1|}.$$

$|\mathrm{Spin}^c(Y)| = |H_1(Y;\mathbb{Z})| = p.$



$$\widehat{HFK}(Y,K,\mathfrak{s}) \cong \begin{cases} \widehat{HFK}(\mathfrak{b}(u,v)) \\ \widehat{HFK}(\mathfrak{b}(u-2v,v)) \end{cases}.$$

Constrained knots

Theorem (Oszváth and Szabó '03)

For any alternating knot $K \subset S^3$, $\widehat{HFK}(K)$ (with mod 2 Maslov grading and Alexander grading) is determined by its Alexander polynomial $\Delta_K(t)$.

Remark

For an alternating knot K, coefficients of $\Delta_K(t)$ are alternating. Hence

$$|\Delta_K(-1)| = u \text{ for } K = \mathfrak{b}(u, v).$$

Constrained knots

Summary

- Compare $|\Delta_{K_i}(-1)|$. We have $u_1 = u_2, u_1 - 2v_1 = u_2 - 2v_2$;
- Compare numbers of spin^c structures with $|\Delta_{K_i}(-1)| = u$. We have $l_1 = l_2$;
- Remain to compare $K_1 = C(p, q, l, u, v)$ and $K_2 = C(p, q^{-1}, l, u, v)$.

Remark

- $[K_i] \neq 0 \in H_1(L(p, q^{-1}); \mathbb{Z}) \cong \mathbb{Z}_p$;
- For p prime, compare $[K_1]$ and $[K_2]$. We have $l \in \{2, p\}$.

Constrained knots

Idea of sufficient part: construct an isomorphism of $\pi_1(Y - N(K_i))$.

Theorem (Waldhausen '68)

Suppose $M_i (i = 1, 2)$ are Haken manifolds that are knot complements of K_i. If there is an isomorphism $\psi : \pi_1(M_1) \to \pi_1(M_2)$ that sends meridian to meridian, longitude to longitude, then K_1 and K_2 are equivalent.

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Instanton knot homology

For a knot K in a 3-manifold Y with $[K] = 0 \in H_1(Y;\mathbb{Q})$, Kronheimer and Mrowka define a vector space $KHI(Y,K)$ over $\mathbb{C}$ called **instanton knot homology**. The definition is based on sutured manifolds studied by Gabai, Juhász, *et al.* For gradings, Kronheimer and Mrowka, and then Zhenkun Li, study the $\mathbb{Z} \oplus \mathbb{Z}_2$ grading on KHI by Seifert surface of K. Baldwin and Sivek study the naturality of KHI.

Conjecture (Kronheimer and Mrowka '10)

For a knot K in a 3-manifold Y with $[K] = 0 \in H_1(Y;\mathbb{Q})$, we have

$$KHI(Y,K) \cong \widehat{HFK}(Y,K) \otimes \mathbb{C}.$$

Instanton knot homology

Theorem (Oszváth and Szabó '04 for $\widehat{HFK}$, Lim '09, Kronheimer and Mrowka '10 for KHI)

For a knot K in S^3, graded Euler characteristics $\chi(\widehat{HFK}(K))$ and $\chi(KHI(K))$ both equal to the Alexander polynomial $\Delta_K(t)$ (up to sign).

Remark

From the grading, we have $KHI(Y,K) = \bigoplus_{i\in\mathbb{Z}_2, j\in\mathbb{Z}} KHI_i(Y,K,j)$. The graded Euler characteristic $\chi(KHI(Y,K))$ is defined by

$$\sum_{j\in\mathbb{Z}} (\dim KHI_0(Y,K,j) - \dim KHI_1(Y,K,j)) \cdot t^j.$$

Instanton knot homology

Theorem (Li and Y. '21)

For a knot K in a 3-manifold Y with $\mathrm{Tors} H_1(Y - N(K), \mathbb{Z}) = 0$, we have

$$\chi(KHI(Y,K)) = \chi(\widehat{HFK}(Y,K)) \text{ (up to sign).}$$

Remark

- By work of Friedl, Juhász, and Rasmussen, the right hand can be calculated by $\pi_1(Y - N(K))$ (related to Turaev torsion).
- The homology condition is because KHI doesn't have a decomposition with respect to torsion spin^c structures.

Instanton knot homology

$\chi(KHI(Y,K))$ provides a lower bound of $\dim KHI(Y,K)$. For the upper bound, we have the following theorem.

Theorem (Li and Y. '20, Baldwin, Li, and Y. '20)

For a $(1,1)$ knot K in $Y = S^3$ or $Y = L(p,q)$, we have

$$\dim KHI(Y,K) \leq \dim \widehat{HFK}(Y,K).$$

Remark

In general, we show $\dim KHI(Y,K) \leq \dim \widehat{CFK}(Y,K)$. For $(1,1)$ knots,

$$\dim \widehat{HFK}(Y,K) = \dim \widehat{CFK}(Y,K) = |\alpha \cap \beta|.$$

Instanton knot homology

For any constrained knot $K \subset Y$ with $H_1(Y - N(K), \mathbb{Z}) \cong \mathbb{Z}$, we know $\widehat{HFK}(Y, K)$ is totally determined by $\Delta_K(t)$.

Corollary

For a constrained knot K with $H_1(Y - N(K), \mathbb{Z}) \cong \mathbb{Z}$, we have

$$\dim KHI(Y, K) = \dim \widehat{HFK}(Y, K).$$

Remark

For $K \subset Y = L(p, q)$, we have $H_1(Y - N(K), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_d$ for $d \mid p$.
In progress: remove the homology condition.

Instanton knot homology

Definition

A rational homology sphere Y is called an L-space if

$$\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|.$$

This is a generalization of lens spaces. A knot $K \subset Y$ is called an L-space knot if a Dehn surgery on K gives another L-space.

Theorem (Oszváth and Szabó '05 for $Y = S^3$, J. Rasmussen and S. D. Rasmussen '17 for general Y)

For any L-space knot K in Y with $H_1(Y - N(K), \mathbb{Z}) \cong \mathbb{Z}$, we know $\widehat{HFK}(Y, K)$ is determined by the Alexander polynomial $\Delta_K(t)$.

Corollary

For a $(1,1)$ L-space knot K in S^3 or $L(p,q)$, if $H_1(Y-N(K),\mathbb{Z})\cong\mathbb{Z}$,

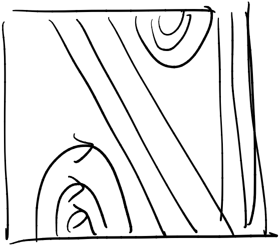
$$\dim KHI(Y,K)=\dim\widehat{HFK}(Y,K).$$

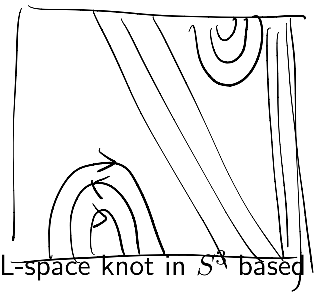
Remark

Torus knots admit lens spaces surgeries (Moser). Torus knots are $(1,1)$ knots. Hence the dimension equation holds for any torus knot.

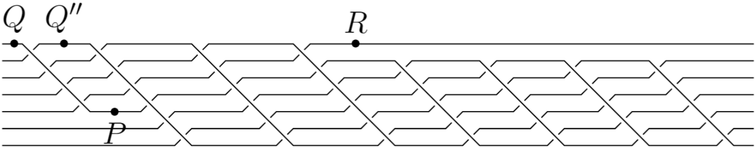
Instanton knot homology

Greene, Lewallen, and Vafaee provide a graphical way to check if a $(1,1)$ knot is an L-space knot.





Recently, Zipei Nie gives a braid diagram of $(1,1)$ L-space knot in S^3 based on the above work.



hyperbolic manifolds

Consider orientable hyperbolic manifold M with $\partial M = T^2$. **Snappy** program provides a list of simple hyperbolic manifolds (with at most 9 ideal tetrahedra).

$L(p,q)$

Based on Dunfield's census of exceptional fillings, we can verify 21922 (in 59068) manifolds are complements of constrained knots. The full list can be found at https://doi.org/10.7910/DVN/GLFLHI or my homepage faniel.wiki/about/.

hyperbolic manifolds

no m005 not orientable

Name	Slope+(p,q,l,u,v)
$m003$	$(1,0)+(10,3,3,1,0),(-1,1)+(5,4,5,3,1),(0,1)+(5,4,5,3,1)$
$m004$	$(1,0)+(1,0,1,5,2)$
$m006$	$(0,1)+(15,4,2,1,0),(1,0)+(5,3,4,3,1)$
$m007$	$(1,0)+(3,1,2,3,1)$
$m009$	$(1,0)+(2,1,2,5,2)$
$m010$	$(1,0)+(6,5,6,3,1)$
$m011$	$(1,0)+(13,3,3,1,0),(0,1)+(9,4,9,3,1)$
$m015$	$(1,0)+(1,0,1,7,2)$
$m016$	$(0,1)+(18,5,3,1,0),(-1,1)+(19,7,2,1,0)$
$m017$	$(0,1)+(14,3,5,1,0),(-1,1)+(21,8,21,1,0),(1,0)+(7,5,6,3,1)$
$m019$	$(0,1)+(17,5,4,1,0),(1,1)+(11,7,11,3,1),(1,0)+(6,5,5,3,1)$
...	...
$m130$	$(1,0)+(16,3,6,1,0),(0,1)+(16,7,16,3,1)$
$m135$	Not from any constrained knot
...	...

hyperbolic manifolds

Suppose $K = C(p,q,l,u,v) \subset Y$ and $M = Y - \mathrm{int}N(K)$. Recall that if $l = 1$, then K is a connected sum of a 2-bridge knot and a core knot of a lens space. Hence M is not hyperbolic.

Theorem (Y. 20)

If M is Seifert fibered (hence not hyperbolic), then $v = \pm 1$.

Conjecture (Y. 20)

If $l > 1$ and $v \neq \pm 1$, then M is hyperbolic.

Remark

This conjecture holds for $p \leq 10, u < 20$ by calculations based on *SnapPy*.

Thanks for your attention.